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On supersymmetry breaking and the Dijkgraaf-Vafa conjecture

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Abstract: We investigate the Dijkgraaf-Vafa proposal when supersymmetry is broken. We consider $U(N)$ SYM with chiral adjoint matter where the coupling constants in the tree-level superpotential are promoted to chiral spurions. The holomorphic part of the low-energy glueball superpotential can still be analyzed. We compute the holomorphic supersymmetry breaking contributions using methods of the geometry underlying the $\mathcal{N} = 1$ effective gauge theory viewed as a Whitham system. We also study the change in the effective glueball superpotential using perturbative supergraph techniques in the presence of spurions.

Keywords: Supersymmetry and Duality, Supersymmetry Breaking, Supersymmetric gauge theory, Matrix Models.

Contents

JHEP03(2006)104

1. Introduction

Dijkgraaf and Vafa have proposed that the low-energy glueball effective superpotential of $\mathcal{N}=1$ supersymmetric gauge theories in four dimensions can be computed via an auxiliary matrix model [1]. The simplest case is a $U(N)$ gauge theory coupled to a massive adjoint chiral matter multiplet Φ with a tree–level superpotential $W(\Phi)$. The proposal stems from a set of string dualities in the framework of geometrically engineered gauge theories, topological strings and matrix models [2, 3, 1]. The large-N matrix model analysis brings in an algebraic curve which may correspond to a Calabi–Yau dual geometry [2]. We shall consider gauge theories that can be obtained from string theories that lead to such geometries. The DV proposal has been tested and supported directly on the field theoretical side by perturbative computation via superfields formalism [4] and then by using arguments based on anomaly equations [5].

We study here the case where susy is broken explicitly (softly and/or non softly) by the introduction of spurionic fields [6]. Holomorphy at large is lost, but holomorphic quantities such as the glueball superpotential can be still analyzed and one can compare the computation in the superfields formalism adapted to spurion fields with that one using the algebraic curve underlying the effective gauge theory. In order to discuss such breaking we utilize two notions: a closed string realization of the method of the spurions [7] and Whitham deformations [8, 9].

It is worth to recall the geometrical origin of such gauge theories for type IIB string theory in order to insert the notion of spurion in a natural way in this language. We have in mind D-branes partially wrapped over non trivial 2-cycles of non compact CY and the dual description where D-branes have been replaced by fluxes [2]. In the UV, adjoint chiral multiplets Φ arise from holomorphic deformations of the supersymmetric cycles and of open string gauge bundles on these cycles. A four-dimensional superpotential for these fields can arise and can be written as $W = W(\Phi, g_k)$ where g_k depend only on the complex structure. From the perspective of the $D3$–brane action in the low–energy limit, where supergravity decouples, the g_k can be interpreted as couplings. As already suggested in [7], the susy breaking parameters are described by auxiliary components of the closed string fields, typically magnetic fluxes along CY directions, depending on the complex structure moduli. Such fluxes are introduced by hand without back reaction of the string or of the supergravity backgrounds. In the four dimensional supergravity language they are F–components of chiral multiplets which depend only on the complex structure moduli. Vev of such F–terms cause spontaneous breaking of local susy and, in the appropriate flat limit with decoupling of supergravity, they appear as explicit breaking terms which can be written in the spurionic fashion in the rigid susy action.

A non–perturbative analysis of susy broken effective dynamics has been done in [10] for $\mathcal{N} = 2$ supersymmetric gauge theories. In that context the connection between the Seiberg–Witten solution [11] and integrable systems (Whitham hierarchy) [8] was used. The authors of [10] break susy promoting the Whitham parameters of the hierarchy to spurions and then compute the broken effective potential using the $\mathcal{N} = 2$ integrable structure.

As in the $\mathcal{N} = 2$ case, a relation between the Whitham systems and the $\mathcal{N} = 1$ effective geometry was established in [9]. This suggests to break supersymmetry promoting the Whitham parameters to spurions as in the $\mathcal{N} = 2$ case. In the $\mathcal{N} = 1$ geometry the Whitham parameters are precisely the tree–level coupling costants of the matter superpotential [9]. We will break the $\mathcal{N} = 1$ supersymmetry promoting them to spurions, and the Whitham hierarchy can then be interpreted as a family of supersymmetry breaking deformations of the original theory. Using this interpretation, we will compute directly from the geometrical data the holomorphic supersymmetry breaking contributions in the low–energy effective glueball superpotential.

We have also analyzed with perturbative supergraph techniques the effective glueball superpotential when susy is broken with spurions. Arguments for the computability of the effective superpotential have been presented in [7]. If supersymmetry is broken, holomorphicity in the coupling constants is no longer guaranteed, the computation is much harder than in the $\mathcal{N} = 1$ case and the simplifications of [4] do not work in general. Anyway, we can restrict ourselves to a particular subclass of contributions for which a spurionic superfields generalization of the techniques in [4] can be done. Within such strong approximation and with unbroken $U(N)$ gauge group, we find that to all order in the glueball superfield the effective superpotential has the same functional form of the $\mathcal{N} = 1$ case where the coupling constants are replaced by spurions and so it results still holomorphic.

JHEP03(2006)104

The paper is organized as follows: in section 2 we review the geometry underlying the Dijkgraaf–Vafa proposal. In section 3 we introduce supersymmetry breaking by spurions and discuss the low–energy glueball superpotential. In section 4 we discuss the geometry as a Whitham system and use it in the susy broken case. In section 5 we treat the explicit example of a deformed susy broken cubic tree–level superpotential. In section 6 we use perturbative superspace techniques along the line mentioned above. Section 7 is devoted to conclusions. At the end, in two appendices A and B, we describe the computational details of section 5 and section 6.

2. The geometrical picture

We consider the particular case of a $\mathcal{N} = 1$, $U(N)$ gauge theory with a degree $n+1$ polynomial tree–level superpotential $W(\Phi)$ for the chiral matter superfields in the adjoint representation of the gauge group

$$W(\Phi) = \sum_{k=1}^{n+1} \frac{g_k}{k} \mathrm{Tr}\, \Phi^k \,. \tag{2.1}$$

In a generic vacuum the gauge group $U(N)$ is broken to $U(N_1) \times \ldots \times U(N_n)$. In the IR limit the effective low–energy degrees of freedom are described by the glueball superfields $S_i = \frac{1}{32\pi^2} \mathrm{Tr}\, W_i^\alpha W_{\alpha i}$ where W_i^α is the fermionic chiral superfield, field strength of the vector multiplet of the unbroken gauge group $U(N_i)$.

The expression for the non perturbative glueball superpotential reads

$$W_{eff}(S_i) = -\sum_{i=1}^{n} \left[N_i \frac{\partial \mathcal{F}}{\partial S_i} + 2\pi i \tau_i S_i \right] , \tag{2.2}$$

where $\mathcal{F}$ is the prepotential which can be computed from the geometrical data [2, 3]. In [1] it has been proposed to reinterpret and compute this prepotential as the free energy of an associated matrix model. In [5, 12] it was also deduced directly on the field theoretical ground using generalized Konishi anomaly equations.

The geometry associated with the low–energy theory is described by a family of genus $g = n-1$ Riemann surfaces and by a meromorphic differential dS

$$y^2 = [W(x)']^2 + f^{(n-1)}(x) \,, \tag{2.3}$$

$$dS = y\, dx = \sqrt{[W(x)']^2 + f^{(n-1)}(x)}\, dx \,. \tag{2.4}$$

The degree $n-1$ polynomial $f^{(n-1)}(x) = \sum_{l=0}^{n-1} f_l x^l$, is associated with the quantum contributions and the coefficients f_l $(l = 0, \cdots, n-2)$ are the moduli of the complex curve; the derivatives of the meromorphic differential (2.4) with respect to the moduli give holomorphic differentials.

A basis of canonical cycles [9, 12] is $\{\alpha^i, \beta_i, \alpha^0, \beta_0\}$, where $i = 2, \ldots, n$, with intersection numbers $(\beta_b \cap \alpha^a = \delta_b^a)$. The cycles are all compact except β_0. We label the cuts starting from the larger real root of the algebraic curve (2.3), so from right to left. The α^i–cycle

JHEP03(2006)104

surrounds counterclockwise the i–th cut while the α^0–cycle encircles all the cuts and then gives the residue at infinity. The dual β_i–cycle ($i = 2, \dots, n$) passes clockwise through the i-th and the first cut, while β_0 goes from the second sheet infinity to the first passing through the first cut. The periods s_i, the parameter t_0 and the conjugated periods are defined as

$$s_i = \oint_{\alpha^i} dS\,, \qquad t_0 = \oint_{\alpha^0} dS = -Res_\infty(dS) = \frac{f_{n-1}}{2g_{n+1}}\,, \tag{2.5}$$

$$\Pi_i = \frac{1}{2}\oint_{\beta_i} dS\,, \qquad \Pi_0 = \frac{1}{2}\int_{\beta_0} dS\,. \tag{2.6}$$

In these variables the effective superpotential computed by the geometry is

$$-W_{eff} = N\Pi_0 + \sum_{i=2}^{n} N_i\Pi_i = N\frac{\partial\mathcal{F}}{\partial t_0} + \sum_{i=2}^{n} N_i\frac{\partial\mathcal{F}}{\partial s_i}\ , \tag{2.7}$$

where $\sum_{j=1}^{n} N_j = N$. In the previous formula we have introduced the prepotential[1] $\mathcal{F}$ such that its derivatives w.r.t. the $\{s_i, t_0\}$ periods give the dual ones $\{\Pi_i, \Pi_0\}$.

Upon getting the superpotential as a function of the variables s_i and t_0, we return to the variables of [1, 2] using[2]

$$\begin{aligned} s_i &= -2S_i\ , \qquad i = 2, \dots, n\ , \\ t_0 &= -2\sum_{j=1}^{n} S_j\ , \end{aligned} \tag{2.8}$$

in fact the S_i are the physical variables which are interpreted as the glueball superfields.

3. Supersymmetry breaking

The introduction of spurionic fields provides the standard mechanism for the explicit (soft and/or non soft) breaking of global supersymmetry. In the $\mathcal{N} = 1$ case the tree–level superpotential W_{tree}, and the effective glueball prepotential $\mathcal{F}$, depend on the coupling constants g_m associated with the operators $\mathrm{Tr}\,\Phi^m$ in the ultraviolet action. In order to break $\mathcal{N} = 1$ supersymmetry down to $\mathcal{N} = 0$ we promote the coupling constants g_m to $\mathcal{N} = 1$ chiral superfields G_m and then we freeze the scalar and the auxiliary F–components to constant values. In this way the chiral spurions $G_m = g_m + \theta^2\Gamma_m$ produce non supersymmetric terms in the superpotential W_{tree}. We want to study their effects on the low energy glueball effective superpotential under the assumption that the low energy degrees of freedom are still the glueballs. The breaking parameters Γ_m must be considered the smallest scales in the theory. They are thought as small perturbations of the $\mathcal{N} = 1$ theory by keeping fixed the $\mathcal{N} = 1$ vacuum structure and the gauge symmetry breaking patterns $U(N) \to U(N_1) \times \cdots \times U(N_n)$.

[1]The prepotential differs from the usual one [1, 2] for a multiplicative factor due to the change of variables. Anyway, this difference is not felt by the effective superpotential W_{eff} which is a function of the dual periods, the quantities which really enter in the computation, as in (2.7).

[2]The variables of [1, 2] are $S_j = -\frac{1}{2}\oint_{A^j} dS$ and $\Pi_j = \frac{1}{2}\oint_{B_j} dS$, with $\{A_j, B^j;\ j = 1, \dots, n\}$ a different set of cycles with all B_j non compact.

JHEP03(2006)104

We set the scalar components of G_m equal to the coupling constants g_m for $m \leq n+1$, zero for $m > n+1$, and the F–components Γ_m will be considered as small susy breaking parameters for all G_m. Explicitly

$$\begin{aligned} G_k &= g_k + \theta^2 \Gamma_k\,, \qquad && k \leq n+1\ , \\ G_j &= \theta^2 \Gamma_j\,, && j > n+1\ , \end{aligned} \tag{3.1}$$

and hence we will consider tree–level superpotential (2.1) perturbed as

$$W_{tree}(\Phi) = \sum_{k=1}^{n+1} \frac{G_k}{k} \mathrm{Tr}\, \Phi^k + \theta^2 \sum_{j>n+1} \frac{\Gamma_j}{j} \mathrm{Tr}\, \Phi^j\ . \tag{3.2}$$

Notice that besides having promoted to spurion the coupling constants already appearing in the tree–level superpotential, we have also added pure auxiliary F–terms. For $k > 3$ these spurionic terms are not soft and quadratic divergences can appear in the wave function renormalization; in any case they have to be considered as dangerously irrelevant operators with the usual warning [13, 7]. The Γ_m for $m \leq n+1$ can be interpreted as vacuum expectations values of fluxes [7], whereas it is not obvious that this is the case for $m > n+1$. In any case, we will see that the generalization to all the Γ_m terms is of some interest in the application of the Whitham approach.

Let now analize what happens in the effective theory when the Γ_m are turned on. We will restrict ourselves to a discussion of some formal aspects which can be extracted from the geometry of the $\mathcal{N}=1$ case. We assume that in the effective dynamics the emergence of the spurions G_m are controlled by the holomorphic dependence of the $\mathcal{N}=1$ prepotential $\mathcal{F}(S_i, g_m)$ on the coupling constants. If we restrict ourselves to holomorphic terms in the low–energy glueball superpotential, the prepotential in the susy broken phase has the same functional form as the $\mathcal{N}=1$ case where now the coupling constants g_m are replaced by the spurions as G_m. This is essentially a naturalness assumption on the effective superpotential [14]. In section 6 we will discuss these assumptions using superfields perturbative techniques extending [4] to the susy broken case.

We make some comments about the interpretation of the couplings Γ_j ($j > n+1$). They must be understood as coming from tree–level superpotential W_{tree} of degree greater than $n+1$ where also the scalar coupling constants g_j above the $(n+1)$–degree are turned on. The low energy glueball prepotential will also depend on all these couplings. We then consider the effective theory of (3.2) as obtained from that one of higher degree in the limit where $g_j \to 0$ ($j > n+1$) and in the same vacuum of the theory of $(n+1)$–degree[3]. In conclusion the prepotential depends on the n glueball superfields S_i (in our conventions t_0 and s_i) and it is evaluated where $g_j = 0$.

We expand now the prepotential $\mathcal{F}(S_i, G_m)$ around the supersymmetric vacuum. If we consider the case of broken supersymmetry with G_m having the form (3.1), the terms

[3] We can choose $\mathcal{N}=1$ massive theories with classical vacua configuration which are nonsingular for $g_j \to 0$ such that there is analitycity of the glueball superpotential around $g_j = 0$.

with more than one power of Γ_m will not give any contribution and we have

$$\begin{aligned} \mathcal{F}(s_i, g_k, \Gamma_k, \Gamma_j) &= \mathcal{F}(s_i, g_m)|_{g_j=0} \; + \\ &+ \theta^2 \sum_{k=1}^{n+1} \Gamma_k \frac{\partial \mathcal{F}(s_i, g_m)}{\partial g_k}\Big|_{g_j=0} + \theta^2 \sum_{j>n+1} \Gamma_j \frac{\partial \mathcal{F}(s_i, g_m)}{\partial g_j}\Big|_{g_j=0} \; . \end{aligned} \tag{3.3}$$

The first term in this expression is the prepotential of the supersymmetric case for a theory with tree–level superpotential of $(n+1)$–degree. As just discussed the last term is interpreted as coming from an higher degree theory in the appropriate limit.

We now insert this expression in (2.7) and we obtain the holomorphic glueball superpotential associated with a tree–level susy breaking superpotential as (3.2)

$$\begin{aligned} -W_{eff} &= N\left[\frac{\partial \mathcal{F}}{\partial t_0} + \theta^2 \sum_{k=1}^{n+1} \Gamma_k \frac{\partial^2 \mathcal{F}}{\partial t_0 \partial g_k} + \theta^2 \sum_{j>n+1} \Gamma_j \frac{\partial^2 \mathcal{F}}{\partial t_0 \partial g_j}\right] + \\ &+ \sum_{i=2}^{n} N_i \left[\frac{\partial \mathcal{F}}{\partial s_i} + \theta^2 \sum_{k=1}^{n+1} \Gamma_k \frac{\partial^2 \mathcal{F}}{\partial s_i \partial g_k} + \theta^2 \sum_{j>n+1} \Gamma_j \frac{\partial^2 \mathcal{F}}{\partial s_i \partial g_j}\right] . \end{aligned} \tag{3.4}$$

In this expression the first terms within the square bracket are supersymmetric, whereas the others break susy explicitly: they involve second derivatives of the prepotential evaluated where the $g_j \equiv 0$ $(j > n+1)$. We will show in the next section how to obtain directly and efficiently from the geometrical data of the $\mathcal{N}=1$ theory the mixed second derivatives of $\mathcal{F}$ appearing in (3.4) in order to extract the effective supersymmetry breaking contributions.

4. The $\mathcal{N}=1$ geometry and Whitham systems

The geometry of the $\mathcal{N}=1$ low–energy effective theory is associated with the generating meromorphic differential dS (2.4) and it can be thought as coming from a Seiberg–Witten geometry of a $\mathcal{N}=2$ theory [3]. The addition of a superpotential together with a geometric transition and a desingularization leads to such geometry with parameters g_k and complex moduli f_l [2]. The couplings g_k can be viewed as Whitham deformations of the previous SW geometry. Performing a Whitham deformation mean extending the parameter space of the curve with extra variables [8]. As a consequence of this deformation the moduli of the curve and also the generating differential become functions of these new parameters.

As shown in [9] the $\mathcal{N}=1$ geometry can be embedded into the Whitham framework. The moduli f_l of the curve (2.3) are functions $f_l = f_l(g_k, t_0, s_i)$ of the Whitham parameters g_k and of $(t_0,\, s_i)$, the periods of the generating differential dS along the α–cycles. We review some results of [9] and set up our conventions.

One of the advantages we gain using Whitham description is that it provides an efficient way to compute the mixed second derivatives appearing in (3.4) directly in terms of geometrical data since the coupling constants are considered as independent parameters.

Using the whole set of variables $(g_k,\, t_0,\, s_i)$ characterizing the curve (2.3) and the generating differential (2.4), the Whitham system can be defined by the following set of equations [9]

$$\frac{\partial dS}{\partial s_i} = d\omega_i \ , \qquad \frac{\partial dS}{\partial t_0} = d\Omega_0 \ , \qquad \frac{\partial dS}{\partial g_k} = d\Omega_k \ , \tag{4.1}$$

where $d\omega_i$ are normalized holomorphic differentials

$$\oint_{\alpha_i} \frac{\partial dS}{\partial s_j} = \oint_{\alpha_i} d\omega_j = \delta_{ij} \ . \tag{4.2}$$

The differentials $d\Omega_k$ are meromorphic of the second kind with poles only at the infinity points $\pm\infty$; $d\Omega_0$ is a differential of the third kind with residue at $\pm\infty$. They have vanishing α–periods and behave at infinity as ($\xi = \frac{1}{x}$)

$$\oint_{\alpha^i} d\Omega_0 = \frac{\partial s_i}{\partial t_0} = 0 \ , \quad \oint_{\alpha^i} d\Omega_k = \frac{\partial s_i}{\partial g_k} = 0 \ ; \qquad d\Omega_l = -(\xi^{-l-1} + O(1))d\xi \ . \tag{4.3}$$

These normalization conditions characterize s_i, t_0 and g_k as independent variables. The generating differential dS is then a linear combination of the differentials (4.1)

$$dS = \sum_{i=2}^{n} s_i \, d\omega_i + t_0 \, d\Omega_0 + \sum_{k=1}^{n+1} g_k \, d\Omega_k \ = \sqrt{[W(x)']^2 + \sum_{k=0}^{n-2} f_k x^k + 2g_{n+1} t_0 x^{n-1}} \, dx \ . \tag{4.4}$$

Consinstency of the equality in (4.4) requires that

$$g_k = -Res_{\infty+}(x^{-k} dS) \ , \tag{4.5}$$

which can be verified [9]. Using (4.4) the meromorphic differentials $d\Omega_l$ can be written as

$$\begin{aligned} d\Omega_0 &= \frac{\partial dS}{\partial t_0} = \frac{g_{n+1}x^{n-1}}{y}dx + \frac{1}{2}\sum_{l=0}^{n-2}\frac{\partial f_l}{\partial t_0}\frac{x^l}{y}dx \ , \\ d\Omega_k &= \frac{\partial dS}{\partial g_k} = \frac{W'(x)x^{k-1}}{y}dx + \frac{1}{2}\sum_{l=0}^{n-2}\frac{\partial f_l}{\partial g_k}\frac{x^l}{y}dx \, , \quad k = 1, \dots, n \ , \\ d\Omega_{n+1} &= \frac{\partial dS}{\partial g_{n+1}} = \frac{[W'(x)x^n + t_0 x^{n-1}]}{y}dx + \frac{1}{2}\sum_{l=0}^{n-2}\frac{\partial f_l}{\partial g_{n+1}}\frac{x^l}{y}dx \ . \end{aligned} \tag{4.6}$$

In this framework, the prepotential $\mathcal{F}$ and so the special geometry can be introduced thanks to the Riemann bilinear relations which guarantee the integrability condition of the prepotential [9]. We must define correctly the first derivatives of $\mathcal{F}$ with respect to both the periods and the coupling constants

$$\frac{\partial \mathcal{F}}{\partial s_i} = \Pi_i = \frac{1}{2}\oint_{\beta_i} dS \ , \quad \frac{\partial \mathcal{F}}{\partial t_0} = \Pi_0 = \frac{1}{2}\int_{\infty-}^{\infty+} dS \ , \quad \frac{\partial \mathcal{F}}{\partial g_k} = Res_{\infty+}\left(\frac{x^k}{k}dS\right) \ . \tag{4.7}$$

As we have seen in the previous section, the supersymmetry breaking contributions appearing in the effective glueball superpotential (3.4) are mixed second derivatives of the prepotential with respect to the Whitham parameters g_k. Starting from the expressions (4.7) it results

$$\frac{\partial^2 \mathcal{F}}{\partial s_i \partial g_k} = Res_{\infty+}\left(\frac{x^k}{k}d\omega_i\right) \ , \qquad \frac{\partial^2 \mathcal{F}}{\partial t_0 \partial g_k} = Res_{\infty+}\left(\frac{x^k}{k}d\Omega_0\right) \ . \tag{4.8}$$

The right hand side of these formulae express the susy breaking contributions in (3.4) as geometrical quantities which can then be read directly as residues. Nevertheless we have to remind the interpretation of the mixed second derivatives appearing in (3.4). As already mentioned, they should be thought to come from an appropriate higher degree system taking $g_j \to 0$ $(j > n+1)$, with the genus of the curve and (t_0, s_i) kept fixed. Using (4.8), the residues can be computed directly with $g_j = 0$; therefore, $\frac{\partial^2 \mathcal{F}}{\partial t_0 \partial g_j}|_{g_j=0}$ and $\frac{\partial^2 \mathcal{F}}{\partial s_i \partial g_j}|_{g_j=0}$ can be properly obtained from the curve of $(n+1)$–degree which depend only on the couplings g_k, $k = 1, \cdots, n+1$. We can then extract all the mixed second derivatives, included those with respect to g_j, using the $(n+1)$–degree geometry.

This simplification is one of the advantages of the embedding of the geometry in the Whitham framework. With this approach we compute the holomorphic supersymmetry breaking terms in the effective glueball superpotential corresponding to a non–supersymmetric perturbation of the $(n+1)$–degree tree–level superpotential (3.2), without the explicit knowledge of the prepotential $\mathcal{F}$.

5. Tree–level cubic superpotential

We consider the simple case of a supersymmetric $U(N)$ gauge theory with tree–level superpotential

$$W_{tree}(\Phi) = \frac{m}{2}\mathrm{Tr}\Phi^2 + \frac{g}{3}\mathrm{Tr}\Phi^3 \,. \tag{5.1}$$

As suggested before we break supersymmetry promoting the coupling constants of the tree–level superpotential to spurions (3.1), deforming (5.1) as

$$W_{tree}(\Phi) = \frac{m + \theta^2\Gamma_2}{2}\mathrm{Tr}\,\Phi^2 + \frac{g + \theta^2\Gamma_3}{3}\mathrm{Tr}\,\Phi^3 + \theta^2 \sum_{j>3} \frac{\Gamma_j}{j}\mathrm{Tr}\,\Phi^j \,. \tag{5.2}$$

The geometry of the $\mathcal{N} = 1$ solution is described by the following complex curve of genus one with meromorphic differential

$$y^2 = g^2(x - a_1)^2(x - a_2)^2 + f_0 + f_1 x \,, \tag{5.3}$$

$$dS = y\; dx = \sqrt{g^2(x - a_1)^2(x - a_2)^2 + f_0 + 2gt_0 x}\, dx \,. \tag{5.4}$$

with $a_1 = 0$ and $a_2 = -\frac{m}{g}$ the classical roots.

We do the computation of the supersymmetry breaking parts as a series with small width of the cuts and then small values of s_i and t_0. The approach is the same as in [2]. In particular, we have considered the case of classical susy vacua with unbroken gauge group and also the case with $U(N) \to U(N_1) \times U(N_2)$ gauge symmetry breaking pattern. Using (4.8), we compute directly the second mixed derivatives of the prepotential, i.e. the susy breaking contributions. The details of the computations are in appendix A. We express our results directly in terms of the physical glueball superfields S_i $(i = 1, \ldots, n)$ using the change of variables (2.8) at the end of the computation. We will write explicitly only the novel supersymmetry breaking contributions to the low–energy glueball superpotential referring the reader to the literature [2, 15] for the well known $\mathcal{N} = 1$ part.

In the case $U(N) \to U(N)$, using (3.4) with superpotential of the form (5.2) we find

$$
\begin{aligned}
-\frac{1}{N} W_{eff} = &-\frac{1}{N} W_{eff}^{\mathcal{N}=1}(S, m, g) \;+ \\
&- \theta^2 \Gamma_2 \left[\frac{S}{m} \left(1 + \sum_{k=1}^{+\infty} \frac{3}{(k+1)!} \frac{\Gamma(\frac{3k}{2})}{\Gamma(\frac{k}{2})} \left(\frac{8 g^2 S}{m^3} \right)^k \right) \right] \;+ \\
&+ \theta^2 \Gamma_3 \left[\frac{S}{g} \left(\sum_{k=1}^{+\infty} \frac{2}{(k+1)!} \frac{\Gamma(\frac{3k}{2})}{\Gamma(\frac{k}{2})} \left(\frac{8 g^2 S}{m^3} \right)^k \right) \right] \;+ \\
&+ \theta^2 \Gamma_4 \left[\frac{m^4}{64 g^4} \sum_{k=2}^{+\infty} \frac{1}{k!} \left((k+1) \frac{\Gamma(\frac{1}{2}(3k-4))}{\Gamma(\frac{1}{2}k)} - 4 \frac{\Gamma(\frac{1}{2}(3k-1))}{\Gamma(\frac{1}{2}(k+1))} \right) \left(\frac{8 g^2 S}{m^3} \right)^k \right] \;+ \\
&- \theta^2 \sum_{j>4} \frac{\Gamma_j}{j} \left[\frac{g}{j!} \left(\frac{\partial^j}{\partial \xi^j} \frac{1}{\sqrt{(g + m\,\xi)^2 + f_0\, \xi^4 - 4 g S\, \xi^3}} \right) \Bigg|_{\xi=0} + \right. \\
&\qquad \left. + \frac{m}{2(j-1)!} (1 + Y) \left(\frac{\partial^{j-1}}{\partial \xi^{j-1}} \frac{1}{\sqrt{(g + m\,\xi)^2 + f_0\, \xi^4 - 4 g S\, \xi^3}} \right) \Bigg|_{\xi=0} \right] , \qquad (5.5)
\end{aligned}
$$

Y and f_0 are functions of (S, m, g) whose expressions (A.19, A.25) are given in appendix A.

As a consistency check of our computation and focusing on the spurionic terms Γ_2 and Γ_3, we can compare the previous result with the mixed second derivatives of the $\mathcal{N} = 1$ perturbative prepotential

$$
\mathcal{F} = \frac{S^2}{2} \sum_{k=1}^{+\infty} \frac{1}{(k+2)!} \frac{\Gamma(\frac{3k}{2})}{\Gamma(\frac{k}{2} + 1)} \left(\frac{8 g^2 S}{m^3} \right)^k \;, \qquad (5.6)
$$

derived for the first time in [16] from the large-N matrix model. We find a complete agreement except the linear term ($\sim S$) in the series multiplied by Γ_2.

The appearance of the linear term can be explained in the following way. It is known [17, 1, 5] that the measure in the matrix model partition function and also the allowed divergent modes on the complex curve [3] give schematically a contribution like $(S - S \log(m \Lambda_0^2 / S))$ where Λ_0 is a cut-off: this contribution together with the additive term $(2\pi i \tau S)$ in the effective superpotential gives the Veneziano–Yankielowicz superpotential [18]. The derivative of this contribution w.r.t. the coupling m gives exactly the linear term appearing in (5.5) which also agrees with what we have found using perturbative techniques (see section 6).

The supersymmetry breaking part coming from the quartic term (and also from the higher ones) can be checked by comparison with the $\mathcal{N} = 1$ superpotential computed implicitly in [19] for a generic tree–level superpotential. By evaluating the derivative where all the coupling constants except (m, g) are set to zero, we find agreement with their computation for all the finite order explicitly given by them.

JHEP03(2006)104

In the case $U(N) \to U(N_1) \times U(N_2)$, we consider only $\Gamma_2,\ \Gamma_3,\ \Gamma_4$ in (5.2) as source of susy breaking. Then, using (3.4) we have

$$
\begin{aligned}
-W_{eff} \;=\; & -W_{eff}^{\mathcal{N}=1}(S_1,S_2,m,g) \;+ \\
& + \theta^2\Gamma_2\Big[(2N_2-N_1)\frac{S_1}{m}+(2N_1-N_2)\frac{S_2}{m}+30(N_1-N_2)\frac{g^2}{m^4}S_1S_2+ \\
& \quad +\; 3(5N_2-2N_1)\frac{g^2}{m^4}S_1^2+3(2N_2-5N_1)\frac{g^2}{m^4}S_2^2+O(S^3)\Big]+ \\
& + \theta^2\Gamma_3\Big[-2N_2\frac{S_1}{g}-2N_1\frac{S_2}{g}+20(N_2-N_1)\frac{g}{m^3}S_1S_2+ \\
& \quad +\; 2(2N_1-5N_2)\frac{g}{m^3}S_1^2+2(5N_1-2N_2)\frac{g}{m^3}S_2^2+O(S^3)\Big]+ \\
& + \theta^2\Gamma_4\Big[2N_2\frac{m}{g^2}S_1+(2N_1+N_2)\frac{m}{g^2}S_2+\frac{6}{m^2}(2N_1-3N_2)S_1S_2+ \\
& \quad +\; \frac{9}{2}(N_2-2N_1)\frac{S_2^2}{m^2}-\frac{3}{2}(N_1-4N_2)\frac{S_1^2}{m^2}+O(S^3)\Big]\,. \qquad (5.7)
\end{aligned}
$$

where we show terms up to the quadratic order in S; we give in Appendix A a sketch of the computation.

We can check also this case using the results of [3] for a quartic tree–level superpotential. Taking the derivatives of their results with respect to the coupling constants and then making the appropriate limit ($S_3 = 0$ and $g_4 \to 0$) we get exactly our supersymmetry breaking contributions.

Observe that, also in this case, linear terms appear in the supersymmetry breaking series multiplied by Γ's. These can again be understood as coming from the Veneziano–Yankielowicz piece of the effective superpotential. In fact, the scales Λ_i associated with each unbroken gauge group sector $U(N_i)$ are functions of the coupling constants as a consequence of the threshold matching [20]; by taking derivatives w.r.t. the couplings we get exactly those linear contibutions appearing in (5.7).
Finally we note that, up to the quadratic order in $S \equiv S_1$, we can consistently get our first result (5.5) from the second one (5.7) simply by setting ($S_2 = 0,\ N_2 = 0$).

6. Perturbative arguments

In this section we exploit the perturbative approach [4] to discuss, from a field theoretical point of view, our use of the $\mathcal{N} = 1$ prepotential to study the low–energy glueball superpotential in the case with broken susy. We consider only the case of unbroken $U(N)$ gauge group and tree–level superpotential for the adjoint chiral superfield given by $W(\Phi) = \sum_{k=2}^{n+1} \frac{G_k}{k} \mathrm{Tr}\, \Phi^k$ where $G_k = g_k + \theta^2\Gamma_k$ are the spurionic coupling constants.

We recall that, because of holomorphicity, in the $\mathcal{N} = 1$ case the effective superpotential is a function only of the coupling constants g_k and not of the $\overline{g}_k$ [14, 4, 5]. In our case susy is broken by the spurions and holomorphicity in the couplings is not any longer a property of the superpotential.

In a perturbative framework the spurions G_k can be thought as ordinary background chiral superfields. We can then think susy unbroken and the perturbative computations in a superspace approach go using the usual D–algebra [21]. The effective action will be schematically of the form

$$\int d^2\theta d^2\bar{\theta}\, \mathcal{K}(G_k, \overline{G}_k, D^2 G_k, \bar{D}^2 \overline{G}_k, \cdots, S, \overline{S}) + \int d^2\theta\, W_{eff}(G_k, S) + \text{h.c.} \ , \tag{6.1}$$

where the superpotential $W_{eff}(G_k, S)$ is constrained to be a holomorphic function[4] of G_k.

If we choose the particular supersymmetric configuration in which all the chiral superfields G_k are equal to the constants g_k, without any dependence on θ_α, then the $\mathcal{N}=1$ effective superpotential is $W_{eff}(g_k, S)$ and hence its holomorphicity [14].

If we choose instead the configuration $G_k = g_k + \theta^2\Gamma_k$ ($\Gamma_k \neq 0$), we break susy and furthermore we will have two kind of contributions to the glueball superpotential.

The first ones come from $W_{eff}(g_k + \theta^2\Gamma_k, S)$ in (6.1) and are the holomorphic ones we have studied in the previous sections. We call them the holomorphic contributions.

The others are D–terms contributions holomorphic in S but not necessarily in the coupling constants g_k, $\overline{g}_k$, Γ_k and $\overline{\Gamma}_k$ which come from particular contributions to $\mathcal{K}$ in (6.1) and which can be written as $\int d^2\theta$ integrals contributing to the glueball superpotential.[5] These terms in the $\mathcal{N}=1$ case ($\Gamma \to 0$) are zero.

Here we adopt a pragmatic attitude and we study only those contributions to the glueball superpotential which can be computed using the powerful perturbative techniques developed in [4] for the $\mathcal{N}=1$ case.

In [4] the perturbative series was generated using only the propagator of the chiral matter superfield sector and the antichiral superfield $\bar{\Phi}$ was integrated out. This was the central point for their simplifications. In order to be able to integrate out $\bar{\Phi}$ as in [4] we must have interactions only in terms of the chiral superfield Φ and then we consider the following UV action

$$\begin{aligned} S(\Phi,\bar{\Phi}) = &\int d^4x\, d^4\theta\ \mathrm{Tr}\ e^{-V}\bar{\Phi}e^{V}\Phi - \int d^4x\, d^2\theta\ \frac{m}{2}\mathrm{Tr}\,\Phi^2 - \int d^4x\, d^2\bar{\theta}\ \frac{\overline{m}}{2}\mathrm{Tr}\,\bar{\Phi}^2 + \\ &+ \int d^4x\, d^2\theta\ \frac{1}{2}(\theta^2\Gamma_2)\mathrm{Tr}\,\Phi^2 + \int d^4x\, d^2\theta\ \sum_{k=3}^{m}\frac{1}{k}(g_k + \theta^2\Gamma_k)\mathrm{Tr}\,\Phi^k \ , \end{aligned} \tag{6.2}$$

where all the antiholomorphic interactions $\int d^2\bar{\theta}\left[\frac{1}{2}(\bar{\theta}^2\overline{\Gamma}_2)\mathrm{Tr}\,\bar{\Phi}^2 + \sum_{k=3}^{m}\frac{1}{k}(\overline{g}_k + \bar{\theta}^2\overline{\Gamma}_k)\mathrm{Tr}\,\bar{\Phi}^k\right]$ are neglected. Furthermore, since we are interested in the glueball superpotential it is also possible to do the usual simplifications of [4, 5] finding as the relevant action[6]

$$\int d^4x\, d^2\theta \left\{\frac{1}{2\overline{m}}\Phi[\Box - i\mathcal{W}^\alpha\partial_\alpha - m\overline{m}]\Phi + W^{int}_{tree}(\Phi)\right\} \ , \tag{6.3}$$

[4]We are thinking about the case with masses in the Wilsonian approach for which the nonrenormalization argument works without IR patologies.

[5]For example, the reader could think about two terms like (with $W^\alpha = i\bar{D}^2(e^{-V}D^\alpha e^V)$ [21]) $\int d^2\theta d^2\bar{\theta} G(g,\overline{g},\Gamma,\overline{\Gamma},\theta^2)\bar{\theta}^2\overline{\Gamma}S^p = \int d^2\theta G\overline{\Gamma}S^p$ or $\int d^2\theta d^2\bar{\theta} H(g,\overline{g},\Gamma,\overline{\Gamma},\theta^2)\mathrm{Tr}[i(e^{-V}D^\alpha e^V)W_\alpha]\Gamma S^q = \int d^2\theta H\Gamma S^{q+1}$ where G and H are functions of $g,\overline{g},\Gamma,\overline{\Gamma},\theta^2$ and not $\bar{\theta}^2$.

[6]$\mathcal{W}_\alpha = [W_\alpha, \cdots\}$ is the spinorial gauge field strength adapted to the action, as a graded–commutator, on the adjoint representation of the $U(N)$ gauge group.

where W_{tree}^{int} in our susy broken case consists in the second line of (6.2). The difference with respect to [4] is that the tree–level superpotential is now defined in terms of spurionic coupling constants.

Now, from (6.3), it is clear that the glueball superpotential we are going to compute will be holomorphic in S and in all the coupling constants except, at most, for the mass. In particular we observe (we refer to Appendix B for the details) that we can have contributions only of the following form

$$\int d^2\theta \left\{ W_{eff}(G_k, S) + \frac{1}{\overline{m}^2} \sum_l \mathcal{B}_l(g_k, \Gamma_k, \theta) S^l \right\} \quad . \tag{6.4}$$

$W_{eff}(g_k + \theta^2\Gamma_k, S)$ is the holomorphic contribution we have already defined. Instead, the second part of (6.4) is a particular subclass of the D–term contributions discussed before where $\mathcal{B}_l$ are holomorphic in all g_k, Γ_k and possibly depend also on θ^2.

A careful perturbative analysis of (6.4) shows that all the coefficients $\mathcal{B}_l = 0$ vanish $\forall l$ and that the contributions to the glueball superpotential we are computing have the following form

$$\int d^2\theta \left[N\theta^2\Gamma_2 \frac{S}{m} + N\frac{\partial \mathcal{F}_0}{\partial S} \right] \qquad \text{with} \qquad \mathcal{F}_0 = \sum_l \mathcal{F}_{0,l}(g_k + \theta^2\Gamma_k) S^l \ . \tag{6.5}$$

We refer the interested reader to Appendix B for the technical details of our perturbative computations.
In (6.5) $\mathcal{F}_{0,l}$ are the planar amplitudes with l index loops of the dual matrix model [1, 4] where the coupling constants are in this case the spurions $G_k = g_k + \theta^2\Gamma_k$. The first term in (6.5) is given by a 1–loop diagram with one vertex $\frac{1}{2}\theta^2\Gamma_2 \mathrm{Tr}\, \Phi^2$. This term is associated with the 1–loop matter contribution to the Wilsonian beta function for the gauge kinetic term which is implicit in the nonperturbative Veneziano–Yankielowicz superpotential. In the previous section we have seen that this term is also given by the geometrical methods.

We conclude that, within our stringent approximations and in the case of unbroken $U(N)$, the effective glueball superpotential in the presence of spurions (6.5) can still be deduced from the $\mathcal{N} = 1$ holomorphic superpotential supporting the results of the previous sections.

7. Conclusions

The Dijkgraaf–Vafa conjecture with supersymmetry breaking is the subject of this work. We have considered the simple case of $U(N)$ gauge theory with massive adjoint chiral matter multiplet with a polynomial tree–level superpotential. We have studied the case where supersymmetry is broken in the tree–level superpotential by promoting the coupling constants to chiral spurions. We have considered their F–components as non–supersymmetric small perturbations of the $\mathcal{N} = 1$ gauge theory and we have discussed how holomorphy can still play a role. The non–supersymmetric holomorphic contributions to the effective low–energy glueball superpotential have been derived with geometrical methods embedded in the Whitham framework as well as with techniques of superfield formalism with spurionic fields.

JHEP03(2006)104

Non–holomorphic D–terms, soft breaking via gaugino mass, low energy vacua are open to investigation. This goes beyond the information encoded in the holomorphic matrix model that we have used so far.

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A. Solution of the broken superpotential

In this appendix we show the main tools and details used for the computation in the cubic tree–level superpotential of section 5.

We have already written the genus one Riemann surface characterising the solution (5.3), (5.4). We observe that there is one holomorphic differential $\frac{dx}{y}$ defined on this surface. This differential can be expanded around the point at infinity in powers of $\xi = 1/x$

$$\frac{dx}{y} = \sum_{k=0}^{\infty} R_k \xi^k d\xi \ , \qquad R_k = \frac{-1}{k!}\frac{\partial^k}{\partial \xi^k}\left(\frac{1}{\sqrt{(g+m\ \xi)^2 + f_0\ \xi^4 + 2gt_0\ \xi^3}}\right)\Bigg|_{\xi=0} \ , \tag{A.1}$$

where R_m are functions of g_k, t_0, f_0 and can be simply computed by power expansion of y. The normalized holomorphic differential $d\omega$ is then

$$d\omega = \frac{1}{h_0}\frac{dx}{y} \ , \tag{A.2}$$

where we have introduced the following quantities[7]

$$h_m = \oint_\alpha \frac{x^m dx}{y} \, . \tag{A.3}$$

The meromorphic differentials $d\Omega_k$ are defined by[8]

$$d\Omega_0 = \frac{\partial dS}{\partial t_0} = g\frac{xdx}{y} + \frac{1}{2}\frac{\partial f_0}{\partial t_0}\frac{dx}{y} \, , \qquad d\Omega_k = \frac{\partial dS}{\partial g_k} \qquad k = 2, 3 \, , \tag{A.4}$$

and are completely fixed by the normalization constraints

$$\oint_\alpha d\Omega_0 = 0 \ , \qquad \oint_\alpha d\Omega_k = 0 \qquad k = 2, 3 \, . \tag{A.5}$$

Then $d\Omega_0$ results to be

$$d\Omega_0 = \left(gx - g\frac{h_1}{h_0}\right)\frac{dx}{y} \, , \tag{A.6}$$

[7] The α–cycle encircle counterclokwise the second cut accordingly to our conventions.

[8] We denote $g_2 = m$ and $g_3 = g$ of (5.1).

where we used the first normalization condition of (A.5) which implies $\frac{\partial f_0}{\partial t_0} = -2g\frac{h_1}{h_0}$. Collecting these formulas we can express the second derivatives of the prepotential characterizing the susy breaking terms in the effective superpotential (3.4) for the case under consideration as follow

$$\frac{\partial^2 \mathcal{F}}{\partial t_0 \partial g_k} = Res_0 \left(\frac{\xi^{-k}}{k} d\Omega_0 \right) = \frac{1}{k} \left(gR_k - g\frac{h_1}{h_0} R_{k-1} \right) , \tag{A.7}$$

$$\frac{\partial^2 \mathcal{F}}{\partial s_2 \partial g_k} = Res_0 \left(\frac{\xi^{-k}}{k} d\omega \right) = \frac{R_{k-1}}{k} \frac{1}{h_0} , \tag{A.8}$$

where the R_k are defined in (A.1).

In the case of unbroken gauge group ($U(N) \to U(N)$) the cut associated with the s_2 variable degenerate to a point with $s_2 \to 0$ and the only variable is t_0. The curve (5.3) can be written as

$$y^2 = g^2(x - x_1)^2(x - x_3)(x - x_4) , \tag{A.9}$$

where $x_3,\ x_4$ are the extremal points of the first cut and x_1 is the double zero of the curve where the second cut degenerates. Following [2] it is useful to introduce the quantities

$$\Delta_{43} = \frac{1}{2}(x_4 - x_3) \quad , \quad \Delta = (a_1 - a_2) = \frac{m}{g} , \tag{A.10}$$

$$Q = \frac{1}{2}(x_4 + x_3 + 2x_1) = (a_1 + a_2) = -\frac{m}{g} , \tag{A.11}$$

$$I = \frac{1}{2}(x_4 + x_3 - 2x_1) = \sqrt{\Delta^2 - 2\Delta_{43}^2} , \tag{A.12}$$

$$x_1 = \frac{Q - I}{2} \quad , \quad \alpha = \frac{g^2}{m^3} \quad , \quad \sigma = 8\alpha S . \tag{A.13}$$

The above relations can be proved comparing (5.3) and (A.9). We have also directly introduced the physically relevant variable $S \equiv S_1 = -\frac{t_0}{2}$.
Being interested in finding $\frac{\partial^2 \mathcal{F}}{\partial g_k \partial t_0}$ as in (A.7) we have evaluated h_1/h_0 in this case

$$\frac{h_1}{h_0} = x_1 = \frac{Q - I}{2} . \tag{A.14}$$

Then we find

$$\frac{\partial^2 \mathcal{F}}{\partial t_0 \partial g_k} = \frac{g}{k} \left(R_k - \frac{h_1}{h_0} R_{k-1} \right) = \frac{gR_k}{k} + \frac{mR_{k-1}}{2k}(1 + Y) , \tag{A.15}$$

where Y is defined as

$$Y = \frac{I}{\Delta} = \sqrt{1 - \frac{2\Delta_{43}^2}{\Delta^2}} . \tag{A.16}$$

Then $I \equiv I(\Delta_{43}^2)$ is a function of Δ_{43}^2. Written in terms of x_i the variable t_0 is

$$\begin{aligned} t_0 = -Res_\infty(dS) &= -Res_\infty[g(x - x_1)\sqrt{(x - x_3)(x - x_4)}\, dx] = \\ &= \frac{g}{16}(2x_1 - x_3 - x_4)(x_4 - x_3)^2 = -\frac{g}{2}\Delta_{43}^2\, I . \end{aligned} \tag{A.17}$$

From (A.12, A.17) we find

$$\sigma = (1 - Y^2)Y\,, \tag{A.18}$$

which gives Y, and then $\frac{\partial^2 \mathcal{F}}{\partial g_k \partial t_0}$, as a function of $\sigma = 8\alpha S$. Solving (A.18) and taking the appropriate branch we obtain

$$Y = \frac{2^{\frac{1}{3}}}{3\left(\sqrt{\sigma^2 - \frac{4}{27}} - \sigma\right)^{\frac{1}{3}}} + \frac{\left(\sqrt{\sigma^2 - \frac{4}{27}} - \sigma\right)^{\frac{1}{3}}}{2^{\frac{1}{3}}} = 1 - \frac{1}{2}\sum_{k=1}^{+\infty} \frac{(8\alpha S)^k}{k!} \frac{\Gamma(\frac{1}{2}(3k-1))}{\Gamma(\frac{1}{2}(k+1))} \tag{A.19}$$

Once R_k ($k = 1, 2, 3$) are found from (A.1) we have the first two softly broken terms of (5.5)

$$\frac{\partial^2 \mathcal{F}}{\partial t_0 \partial m} = -\frac{S}{m}\left[1 + 3\sum_{k=1}^{+\infty} \frac{(8\alpha S)^k}{(k+1)!} \frac{\Gamma(\frac{3k}{2})}{\Gamma(\frac{k}{2})}\right], \tag{A.20}$$

$$\frac{\partial^2 \mathcal{F}}{\partial t_0 \partial g} = \frac{2S}{g}\sum_{k=1}^{+\infty} \frac{(8\alpha S)^k}{(k+1)!} \frac{\Gamma(\frac{3k}{2})}{\Gamma(\frac{k}{2})}\,. \tag{A.21}$$

For the third term of (5.5) the computation is a little more involved. The derivative to be computed is

$$\frac{\partial^2 \mathcal{F}}{\partial t_0 \partial g_4} = \frac{g}{4}R_4 + \frac{m}{8}R_3(1+Y)\,. \tag{A.22}$$

The coefficients R_3 and R_4 are

$$R_4 = \frac{f_0}{2g^3} + \frac{6m}{g^3}S - \frac{m^4}{g^5}\,, \qquad R_3 = \frac{m^3}{g^4} - \frac{2S}{g^2}\,, \tag{A.23}$$

where the unknown function f_0 appears. To compute it we integrate in S the equation that can be obtained from the first constraint in (A.5) and from (A.14)

$$\frac{\partial f_0}{\partial t_0} = -2g\frac{h_1}{h_0} = -2g\frac{Q - I}{2} = m + gI\,. \tag{A.24}$$

We then have

$$f_0[S] = -2mS - 2g\int \frac{m}{g}Y[S]dS = c_1 - 4mS + \frac{m^4}{8g^2}\sum_{j=2}^{+\infty} \frac{(8\alpha S)^j}{j!} \frac{\Gamma(\frac{1}{2}(3j-4))}{\Gamma(\frac{1}{2}j)}\,, \tag{A.25}$$

where c_1 is a function only of the couplings m and g. Using the other constraints in (A.5) that define the derivatives of f_0 with respect to the couplings (m, g) it can be proven that c_1 vanishes. Finally, we use the formulas (A.22, A.23, A.25) with $c_1 = 0$ and find

$$\frac{\partial^2 \mathcal{F}}{\partial t_0 \partial g_4} = \frac{m^4}{64g^4}\sum_{k=2}^{+\infty} \frac{(8\alpha S)^k}{k!}\left((k+1)\frac{\Gamma(\frac{1}{2}(3k-4))}{\Gamma(\frac{1}{2}k)} - 4\frac{\Gamma(\frac{1}{2}(3k-1))}{\Gamma(\frac{1}{2}(k+1))}\right). \tag{A.26}$$

We observe that with these ingredients one has formally all the needed quantities to compute in a closed form, as power series of S, all the susy breaking terms in the effective superpotential (3.4) coming from higher order supersymmetry breaking deformation in the tree–level superpotential (3.2). These are functions only of $Y(S)$ and $f_0(S)$ as shown in (A.15). In fact the coefficient R_k are defined as (A.1) and are functions only of $t_0 = -2S$, of the couplings, and of $f_0(S)$ which is known (A.25). At the end the result is (5.5).

The computation in the case of broken gauge group ($U(N) \to U(N_1) \times U(N_2)$) uses a procedure as [2], making the calculation as a power series in the width of the cuts. Our aim is again to compute the susy breaking terms appearing in (3.4) using the formulas (A.7, A.8). The main difference with the unbroken gauge group case is that now the curve does not degenerate

$$y^2 = g^2(x - x_1)(x - x_2)(x - x_3)(x - x_4) \ . \tag{A.27}$$

We introduce quantities analogous as before[9]

$$\Delta_{43} = \frac{1}{2}(x_4 - x_3) \quad , \quad \Delta_{21} = \frac{1}{2}(x_2 - x_1) \ , \tag{A.28}$$

$$Q = \frac{1}{2}(x_4 + x_3 + x_2 + x_1) = (a_1 + a_2) = -\frac{m}{g} \ , \tag{A.29}$$

$$I = \frac{1}{2}(x_4 + x_3 - x_2 - x_1) = \sqrt{\Delta^2 - 2\Delta_{43}^2 - 2\Delta_{21}^2} \ . \tag{A.30}$$

We don't have anymore the simplification (A.14) and we have to write the integrals h_0 and h_1 as power series in the widths of the cuts ($O(\Delta_{ab}^3)$). We then find the inverse expression of the widths of the cuts Δ_{ab} as a functions of (s_2, t_0) and obtain (h_0, h_1) in terms of (s_2, t_0).

We have also to evaluate the parameters R_k. They have the form (A.1) but now f_0 has to be understood as a function of two variables and $t_0 = -2(S_1 + S_2)$. Precisely f_0 is a function of t_0 and s_2 which are the independent variables and it is determined through the relations

$$\frac{\partial f_0}{\partial t_0} = -2g\frac{h_1}{h_0} \quad , \quad \frac{\partial f_0}{\partial s_2} = \frac{1}{\frac{\partial s_2}{\partial f_0}} = \frac{1}{\oint_\alpha \frac{dx}{2y}} = \frac{2}{h_0} \ . \tag{A.31}$$

The first equation comes from the normalization condition (A.5) while the second one is a consequence of the definition (2.5) of the variable s_2 using the explicit form (5.4) of the differential dS.

We then compute directly the second derivatives of the prepotential, the susy breaking terms (3.4), using (A.7, A.8). At the end of the computation we change variables (2.8) to express the superpotential in terms of the physical glueball superfields S_i. What we find is (5.7).

B. Details on the perturbative approach

In this appendix we explore the details which give (6.5) from (6.2). We will use and extend to spurions the method developed in [4, 22] also reviewing, for reader convenience, their basic steps.

Starting from (6.2), the propagator is the same as in [4]

$$\langle \Phi\Phi \rangle = \frac{-\overline{m}}{\Box - m\overline{m} - i\mathcal{W}^\alpha \partial_\alpha} \ . \tag{B.1}$$

[9]Note that as concern the classical roots there are no modifications.

The gauge field strength is considered constant then the bosonic and fermionic integrations completely decouple in the computation [4]. To compute contributions to the glueball superpotential, we will use the usual chiral–ring properties of W_α [4, 5]. For example, we will use $\mathrm{Tr}(W_\alpha)^n = 0$ with $n > 2$.

Using the double line notation, a Riemann surface (oriented for $U(N)$) with genus g is associated to each topologically relevant diagram with L momentum loop and l index loop, so that $L = l + 2g - 1$. The D–algebra is exactly as in [4]. The only difference is that performing the D–algebra some ∂_α can act on the θ of the spurions $G_k = g_k + \theta^2\Gamma_k$ giving new terms.

It is possible to do some general considerations using the constraints given by the D–algebra structure, the properties of W_α and the geometry of the diagrams in the amplitudes.

We fix a diagram with L momentum loops and $l = L - 2g + 1$ index loops. From the W_α properties it follows that, for a relevant amplitude for the glueball superpotential, the maximal number of allowed W_α is $2l$ otherwise we would have at least one index loop with more than three W_α. Furthermore, in order to perform the fermionic loop integrations it is necessary to have at least $2L$ ∂_α and then at least $2L$ W_α. The number of W_α ($\#W_\alpha$) in a non–trivially zero diagram then satisfies the inequality

$$2L \leq \#W_\alpha \leq 2l = 2L + 2 - 4g \quad . \tag{B.2}$$

This implies that $g \equiv 0$ and the only relevant diagrams to be considered are planar. Moreover the relevant contributions to the glueball superpotential have $\#W_\alpha = 2L, 2L+2$.

We consider first the case $\#W_\alpha = 2L$ i.e. $\#\partial_\alpha = 2L$. In this case the D–algebra has to be done only inside the fermionic loops and then no derivative acts on the background spurions. This is equivalent to say that these contributions are insensible to the θ dependence of the G_k. Then for these kind of terms we can reabsorb the quadratic vertex $\frac{1}{2}\theta^2\Gamma_2 \mathrm{Tr}\,\Phi^2$ into the propagator (B.1) by simply doing the redefinition $m \to m + \theta^2\Gamma_2$. This is clearly true exept a 1–loop amplitude with one vertex $\frac{1}{2}\theta^2\Gamma_2\mathrm{Tr}\,\Phi^2$ contracted with the propagator (B.1) which gives the first term of (6.5). The resulting contribution to the glueball superpotential with $\#W_\alpha = 2L$ is then given by (6.5) and, except the linear term in S, these term are computed perturbatively using the dual matrix model of the $\mathcal{N} = 1$ case.

Now, we consider the other case $\#W_\alpha = \#\partial_\alpha = 2L + 2$. All index loops are now saturated and we have a contribution proportional to $S^{(L+1)}$.

These contributions are nonholomorphic since they are proportional to $\overline{m}^{-2}$. In fact, the corresponding diagrams have a multiplicative factor $\overline{m}^{P_i}$ from the numerator of the P_i propagators (B.1). Expanding the propagators (B.1) at order $(\mathcal{W}^\alpha\partial_\alpha)^{(2L+2)}$, we have $P_f = P_i + 2L + 2$ bosonic propagators $\frac{1}{p^2 + m\overline{m}}$ expressed in momentum space. By redefining the bosonic loop momentum variables $p^2 \to \overline{m}p^2$, from the bosonic Jacobian we are left with a contribution $\overline{m}^{2L}$ while from the denominator of the bosonic propagators we have a term $\overline{m}^{-(P_i+2L+2)}$. Summarizing we have $[\overline{m}^{P_i}][\overline{m}^{2L}][\overline{m}^{-(P_i+2L+2)}] = \overline{m}^{-2}$. Therefore, along the calculation we will set $\overline{m} \equiv 1$ and multiply the final result by $\overline{m}^{-2}$.

In performing the calculation it is convenient to express the propagator (B.1) in the Schwinger variables

$$\int_0^\infty ds_i \ \exp\left[-s_i(p_i^2 + i\mathcal{W}_i^\alpha \partial_\alpha + m)\right]. \tag{B.3}$$

As in [4], the bosonic contribution is given by

$$Z_{boson} = \frac{1}{(4\pi)^{2L}} \frac{1}{(\det M(s))^2}, \qquad M_{ab}(s) \equiv \sum_i s_i L_{ia} L_{ib}, \qquad p_i = \sum_a L_{ia} k_a. \tag{B.4}$$

We note that $M(s)$ is an $L \times L$ matrix and then the denominator of Z_{boson} (B.4) is a homogeneous polynomial of degree $2L$ in s_i. Furthermore, we have, from the fermionic integrations of these diagrams, $2L+2$ $s_i\mathcal{W}_i^\alpha$ terms. Then, at the numerator we have a homogeneous polynomial of degree $2L+2$ in s_i. The degree in s_i of the numerator results to be greater than the denominator degree. Thus, for the class of diagrams with $\#W_\alpha = 2L+2$ (certainly) there is no cancellation between the bosonic and fermionic integrations in contrast with the case $\#W_\alpha = 2L$ [4].

Performing the D–algebra we realize that there are two distinct possibilities depending on the way the two extra ∂_α are distributed on the external spurionic terms. The first possibility is when two ∂_α act on one spurionic constant $\theta^2\Gamma_k$. This contribution would have a multiplicative factor $[(\Gamma_k)\prod_{v=1}^{(V-1)}(g_{k_v} + \theta^2\Gamma_{k_v})]$ (V is the number of vertices of the considered diagram).
The second possibility is when the two ∂_α act on two different spurions. In this case we have a multiplicative term $[(\theta^\alpha\Gamma_k)(\theta_\alpha\Gamma_{k'})\prod_{v=1}^{(V-2)}(g_{k_v} + \theta^2\Gamma_{k_v})]$.

Summarizing the previous considerations, the general structure of the glueball superpotential, due to the integration of the matter fields considering only the holomorphic part of the interaction vertices as in (6.2), is

$$\int d^2\theta \left\{ N\theta^2\Gamma_2 \frac{S}{m} + N\sum_l \mathcal{F}_{0,l}(g_k + \theta^2\Gamma_k) l S^{l-1} + \frac{1}{\overline{m}^2}\sum_l \mathcal{B}_l(g_k, \Gamma_k, \theta) S^l \right\} \quad . \tag{B.5}$$

The $\mathcal{B}_l$ are holomorphic in all g_k, Γ_k and possibly depend also on θ^2. $\mathcal{B}_l$ are analytic in all the variables except m. We will show that $\mathcal{B}_l = 0\ \forall l$ justifying (6.5).

To compute $\mathcal{B}_l$ we must perform the D–algebra and treat the group theoretical factor. As in [4] to simplify the fermionic integrations we can use the fermionic Fourier momentum representation. The novelty in the computation is due to the fact that now there is also the θ^2 from the spurionic vertices to be Fourier transformed. In particular, we have

$$\theta^2 = -\delta^{(2)}(\theta) = -\int d^2\pi e^{i\pi^\alpha\theta_\alpha} \quad . \tag{B.6}$$

We focus on a planar diagram with $L = l-1$ bosonic momentum loops with P propagators and V vertices. In particular we consider the case with only one spurionic constant $\theta^2\Gamma_k$ on which the D–algebra acts nontrivially. The other case is analogue.

The fermionic contribution results to be[10]

$$
\begin{aligned}
Z_{fermion} &= \int \prod_{v=1}^{V} d^2\theta_v\, \theta_1^2 \prod_{i=1}^{P} \left[e^{-s_i \mathcal{W}_i^\alpha i\partial_\alpha} \delta^{(2)}(\theta_{v_i} - \theta_{v_i'}) \right] \\
&= -\int \prod_{i=1}^{P} d^2\pi_i d^2r \prod_{v=1}^{V} d^2\theta_v \prod_{i=1}^{P} \left[e^{-s_i \mathcal{W}_i^\alpha \pi_{i\alpha}} \right] e^{i(\sum_{j_1=1}^{k_1} \pi_{j_1} + r)^\alpha \theta_{1\alpha}} \prod_{v=2}^{V} e^{i(\sum_{j_v=1}^{k_v} \pi_{j_v})^\alpha \theta_{v\alpha}} \\
&= -\int d^2\theta \int \prod_{a=1}^{P-V+2} d^2\kappa_a d^2 r\, \delta^{(2)}\Big(\sum_{j_1=1}^{k_1} \pi_{j_1} + r \Big) \prod_{i=1}^{P} \left[e^{-s_i \mathcal{W}_i^\alpha \pi_{i\alpha}} \right] e^{i(\sum_{j_V=1}^{k_V} \pi_{j_V})^\alpha \theta_\alpha} \\
\Longrightarrow &- \int d^2\theta \int \prod_{a=1}^{l} d^2\kappa_a \prod_{i=1}^{P} \left[e^{-s_i \mathcal{W}_i^\alpha \pi_{i\alpha}} \right] \quad .
\end{aligned}
\tag{B.7}
$$

In the second line π_{j_v} are the spinorial momentum connected to the v–th θ–vertex and satisfy $\pi_{j_v} \equiv \pm\pi_i$ where the sign is $+$ $(-)$ if the spinorial momentum is going outside (inside) the vertex v. In the last two lines we exploit the relations $\sum_{j_v=1}^{k_v} \pi_{j_v} \equiv 0$ $(v = 2, \cdots, V-1)$ with which we define the remaining l spinorial variables κ_a from the independent π_i. Furthermore, in the last line we have used the fact that we are searching for a contribution to $\mathcal{B}_l$ which has $\#\mathcal{W}_\alpha = 2l = 2L+2$. Then, in the expansion of $e^{-i(\sum_{j_V=1}^{k_V} \pi_{j_V})^\alpha \theta_\alpha}$ we can keep only 1, the term independent of θ_α. This is equivalent to say that only the term in which θ^2 of the spurion is killed by two ∂_α contributes to $\mathcal{B}_l$.
At the end of the above manipulations we remain with l fermionic integrations over the indipendent variables κ_a and the π_i are linear combinations of them

$$
\pi_{i\alpha} \equiv \sum_{a=1}^{l} \widetilde{L}_{ia} \kappa_{a\alpha} \quad . \tag{B.8}
$$

As in [4] we can implement the requirement of having two insertions of $\mathcal{W}^\alpha$ for each index loop introducing $2l$ auxiliary grassmanian variables $\mathcal{W}_m^\alpha$ adapted to the action on the adjoint representation with

$$
\mathcal{W}_i^\alpha \equiv \sum_{m=1}^{l} K_{im} \mathcal{W}_m^\alpha \quad . \tag{B.9}
$$

The matrix K is defined so that for each oriented i–th propagator the m–th index loop can coincide and be parallel giving $K_{im} = 1$; or coincide and be anti–parallel giving $K_{im} = -1$; or not coincide giving $K_{im} = 0$.

Summarazing we find from the fermionic integration

$$
\begin{aligned}
&(16\pi^2 S)^l \int \prod_{a,m=1}^{l} d^2\kappa_a d^2\mathcal{W}_m \;\exp\left[-\sum_i s_i \left(\sum_{a,m} \mathcal{W}_m^\alpha K_{mi}^T \widetilde{L}_{ia} \kappa_{a\alpha} \right) \right] \\
&= (16\pi^2 S)^l \int \prod_{a,m=1}^{l} d^2\kappa_a d^2\mathcal{W}_m \;\exp\left[-\sum_{a,m} \mathcal{W}_m^\alpha \widetilde{N}(s)_{ma} \kappa_{a\alpha} \right] \\
&= S^l (4\pi)^{2l} (\det \widetilde{N}(s))^2 \ ,
\end{aligned}
\tag{B.10}
$$

[10]The index j_v, depending on $v = \{1, \cdots, V\}$, runs from 1 to k_v which is the degree of the interaction vertex v as: $\mathrm{Tr}\, \Phi^{k_v}$.

with

$$\widetilde{N}(s)_{ma} \equiv \sum_i s_i K^T_{mi} \widetilde{L}_{ia} \ . \tag{B.11}$$

The relevant fact is that, for our class of diagrams which has an S^2 topology, the matrix K has a nontrivial kernel. In fact, for example, the vector b_m, whose components are all equal to one, belong to the kernel of K_{im}[11]. This is simply due to the fact that in the case we are studying all momentum propagator lines have exact two index loop passing through them with opposite orientation; then, $\forall i$ there will be only one $K_{im'} = 1$ and one $K_{im''} = -1$ ($m' \neq m''$) and $\sum_m K_{im} b_m = 1 - 1 = 0$. It follows that also the matrix $[\widetilde{N}(s)]^T_{am} = \sum_i s_i \widetilde{L}^T_{ai} K_{im}$ has a nontrivial kernel indipendently of the explicit form of $\widetilde{L}$ which we have not analyzed in detail. Then $\det(\widetilde{N}(s)) \equiv 0$. This imply that $\mathcal{B}_l \equiv 0 \ \forall l$ as claimed before.

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